arXiv:hep-th/9404140v2 22 Sep 1994

EXTRINSIC CURVATURE DEPENDENCE OF NIELSEN-OLESEN STRINGS

Peter Orland

The City College of the City University of New York,
New York, NY 10031

and

Baruch College,
The City University of New York,
New York, NY 10010
orlbb@cunyvm.cuny.edu [1]

Abstract

It is shown how to treat the degrees of freedom of Nielsen-Olesen vortices in the $3 + 1$-dimensional $U(1)$ higgs model by a collective coordinate method. In the london limit, where the higgs mass becomes infinite, the gauge and goldstone degrees of freedom are integrated out, resulting in the vortex world-sheet action. Introducing an ultra-violet cut-off mimics the effect of finite higgs mass. This action is non-polynomial in derivatives and depends on the extrinsic curvature of the surface. Flat surfaces are stable if the coherence length is less than the penetration depth. It is argued that in the quantum abelian higgs model, vortex world-sheets are dominated by branched polymers.

---

[1]Permanant Address

1 Introduction

The first topological soliton found in relativistic gauge theory was Nielsen and Olesen's string vortex solution [1]. This solution is essentially similar to the non-relativistic vortex found by Abrikosov [2]. Further work on the structure of non-abelian vortex strings was done subsequently by Ezawa and Tze [3].

Shortly after vortex string solitons were discovered in relativistic gauge theories, Förster [4] and Gervais and Sakita [5] studied their quantum dynamics. Their analyses were performed in the limit that both the penetration depth and coherence length were very small, and concluded that, in that limit, the vortex is described quite well by a Nambu action.

Since Polyakov [6] discussed the addition of terms in the string action depending on the extrinsic geometry of the string world-sheet, a number of people have attempted to determine whether such terms exist for the Nielsen-Olesen vortex [7], [8], [9]. In some early references [7], [8], there was a consensus that there was a rigidity term in the action of the vortex, but the sign of the leading term differed in these two papers. The most recent word in the literature [9] was that no extrinsic curvature dependence seems to be present after all (or if it is present, it is numerically small). The basic method used in references [7], [8], [9] is a version of classical perturbation theory.

The presence of terms quadratic in the extrinsic curvature tensor in the string action [6], [10] do not appear to stabilize the "branched polymer" disease of quantum strings [11]. This is because such terms are irrelevant in the infrared. Polyakov suggested that this may not be so if another term, which counts the number of points in which the world-sheet intersects itself, modulo two, is included in the action (this term had been written down earlier by Balachandran et. al. [12]). It appears that even classically, certain solutions are unstable [13]. Nonetheless, the presence of such terms may have implications for cosmic strings [14], which are essentially classical Nieslen-Olesen vortices (these strings are presumably coherent states with energies far above the ground state). The mechanism for the formation of topological defects proposed by Kibble [14] has recently been verified in the context of liquid crystal systems [15].

In this paper, the world-sheet action of the Nielsen-Olesen vortex is calculated in the london limit, where the higgs mass becomes infinite. The calculation is equally valid for the classical or quantum case. The method is closely related to a representation of the lattice higgs model [16]. It is argued that the effect of a finite higgs mass is taken into account by the introduction of an ultraviolet cut-off. This cut-off is then the reciprocal of the coherence length. The result is an action analogous to that of Rasetti and Regge, Davis and Shellard and of Lund [17] for liquid helium. However, this form of the world-sheet action does not make the extrinsic geometric properties completely clear. The result obtained disagrees with all of the previous references [7], [8] and [9]. To leading order in the extrinsic curvature, the string is found to be anti-rigid (though evidence is found that the action is stable if higher orders are taken into account). The result found in reference [7] was similar, but these authors found another term at leading order, which is not revealed by the analysis here.

Effective action methods for vortex dynamics has been criticized by Arodz and Wegrzyn [18]. They pointed out that integrating out some degrees of freedom leads to a non-local action, which may not correctly capture the features of the original

model. The essential difficulty is due to the fact that green's functions depend on the initial and final conditions. This problem is circumvented in this paper by going to euclidean space, and considering finite string manifolds. The basic results (stability for penetration depth greater than coherence length, no scaling of the string tension in the quantum theory) can then be carried back to Minkowski space.

The action of the vortex in the london limit is non-polynomial in derivatives on the world-sheet (thus non-local). While a proof is still lacking, this action appears to be stable. Small fluctuations about flat surfaces are examined. The action of a long thin tube is calculated, and it is argued that in the quantum theory branched polymers will dominate the world-sheet. This means that the vortex string tension probably does not scale for any choice of couplings of lattice electrodynamics coupled to a higgs field.

2 Collective Coordinates

The starting point is the abelian higgs model in euclidean space, with lagrangian

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\nu + iA_\nu) \phi^* (\partial^\nu - iA^\nu) \phi + \lambda (\phi^* \phi - v^2)^2 \ . \tag{1}$$

As Nielsen and Olesen showed [1], the equations of motion of (1) admit stable vortex string solutions, similar to those of Abrikosov [2], provided that the higgs mass $m_H = 2\sqrt{2}\lambda v$ is greater than the vector boson mass $m_{VB} = ev$. Recall that the coherence length is $\xi = m_H^{-1}$ and the penetration depth is $D = m_{VB}^{-1}$.

It is convenient to parametrize the higgs field ϕ by density and phase variables, similar to those of Tomonaga, Bohm et. al. [19], and Gervais and Sakita [5],

$$\phi(x) = [v + h(x)] \, e^{i \int_{-\infty}^x V_\mu(y) \, dy^\mu} \ . \tag{2}$$

The integral is path dependent, but its exponential is not. The density field h is restricted by $h \geq -v$. The field $V_\mu(x)$ is the superfluid velocity [19]. Under a gauge transformation $\delta A_\mu = \partial_\mu \chi$, $\delta V_\mu = \partial_\mu \chi$ and $\delta h = 0$. This parametrization is essentially the same as that of Gervais and Sakita [5].

The field V_μ is single-valued, but $\int_{-\infty}^x V_\mu(y) \, dy^\mu$ is multi-valued. The line integral of V_μ around a closed path is an integer multiple of 2π; the integer is the (oriented) number of vortex world-sheets the path winds around in four dimensions. This fact expresses $\Pi_1(U(1)) = Z$. If the union of all vortex world-sheets is the (possibly disjoint) oriented two-dimensional manifold Σ, Stokes' theorem implies

$$\partial_\alpha V_\beta - \partial_\beta V_\alpha = 2\pi \int_\Sigma d\sigma^1 \wedge d\sigma^2 \, \delta^4(x - z(\sigma)) \, \epsilon_{\alpha\beta\rho\tau} \frac{\partial z^\rho}{\partial \sigma^1} \frac{\partial z^\tau}{\partial \sigma^2} \, n(\sigma) \ , \tag{3}$$

where σ^1 and σ^2 are the world-sheet (base space) coordinates, z^ρ are the space-time (target space) coordinates and $n(\sigma)$ is the vorticity (which is a constant integer on each connected component of Σ).

Writing the lagrangian (1) in terms of the collective fields (2) and the vortex degrees of freedom (3) gives, with $n(\sigma) = 1$,

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\nu h \partial^\nu h + \lambda h^2 (h + 2v)^2 + \frac{v^2}{2} (A_\mu - V_\mu)(A^\mu - V^\mu)$$

$$+ \quad \frac{1}{2}h(h+2v)(A_\mu - V_\mu)(A^\mu - V^\mu)$$

$$+ \quad \frac{i}{2}\epsilon^{\mu\nu\alpha\beta} A_{\mu\nu} [\partial_\alpha V_\beta - \partial_\beta V_\alpha - \pi \int_\Sigma d\sigma^a \wedge d\sigma^b \, \delta^4(x - z(\sigma)) \, \epsilon_{\alpha\beta\rho\tau} \frac{\partial z^\rho}{\partial \sigma^a} \frac{\partial z^\tau}{\partial \sigma^b}]$$

$$+ \quad i\omega(h+v) \int_\Sigma d\sigma^1 \wedge d\sigma^2 \, \delta^4(x - z(\sigma)) \quad . \tag{4}$$

The new antisymmetric tensor field $A_{\mu\nu}$ and scalar field ω are Lagrange multipliers. This formalism is not convenient for all purposes because of the restriction $h \geq -v$. The last term enforces the condition that $\phi = 0$ on the world-sheet. Notice that if the vortex degrees of freedom, $z(\sigma)$ are integrated out in the functional integral, no new terms depending on the fields h or V_μ are induced in the lagrangian. Thus (4) is completely equivalent to (1). In the quantum theory, the measure on the world-sheet field $z(\sigma)$ must be diffeomorphism invariant and divided by the volume of the diffeomorphism group.

3 The Higgs Mass as a UV Cut-off

The lagrangian (4) is not easy to work with. The chief goal of this article is to integrate out the fields in the target space, leaving behind the action of $z(\sigma)$ in the base space. However this goal is quite difficult with (4). Fortunately the difficulty disappears in the london limit, $\lambda \to \infty$. In the usual way of taking this limit, the penetration depth, D, is fixed, while the coherence length, ξ, tends to zero. Now, if the system is regularized by an ultraviolet cut-off Λ, the coherence length will not vanish as the higgs mass is increased, but instead tend to Λ^{-1}. Thus, by putting such a cut-off into the theory, one does not expect the world-sheet action to be very different from that of the usual abelian higgs model. This observation is similar to that of Kirkman and Zachos, who noted that the energy density away from the core of a magnetic monopole is determined only by one length scale, namely the vector boson mass, if the higgs mass is very large [20] Furthermore, it will become clear that there is a self-energy divergence in the string tension (just as the electromagnetic self-energy of an electron is divergent), which needs to be regularized by a non-zero coherence length.

The lagrangian in the london limit is

$$\begin{aligned}
\mathcal{L} &= \quad \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{v^2}{2}(A_\mu - V_\mu)(A^\mu - V^\mu) \\
&+ \quad i\epsilon^{\mu\nu\alpha\beta} A_{\mu\nu}\partial_\alpha V_\beta - \pi i \, A_{\mu\nu} \int_\Sigma d\sigma^a \wedge d\sigma^b \, \delta^4(x - z(\sigma)) \frac{\partial z^\mu}{\partial \sigma^a} \frac{\partial z^\nu}{\partial \sigma^b} \\
&+ \quad \mu_0 \int_\Sigma d^2\sigma \, \sqrt{det \, g} \quad ,
\end{aligned} \tag{5}$$

where g_{ab} is the induced metric,

$$g_{ab} = e_a \cdot e_b = \sum_\mu e_a^\mu e_b^\mu \quad . \tag{6}$$

and e_a is the vector field

$$e_a^\mu = \partial_a z^\mu(\sigma) \quad . \tag{7}$$

4

The last term is a Nambu action, which comes about because the energy density of the higgs field between the vortex core (where $h = -v$) and the surrounding vacuum region (where $h = 0$) is non-zero. The coefficient of this term, μ_0, depends on the coherence length in a nontrivial way.

4 Duality

The next step is to integrate out everything but $z(\sigma)$ from the functional integral with action (5). There is nothing fancy about doing this. The result is precisely that obtained by eliminating the fields from the equations of motion, since (5) is at most quadratic in the fields in euclidean space. As an optional intermediate step, one can integrate out only A_μ and v_μ leaving behind only the antisymmetric tensor field $A_{\mu\nu}$ as well as z^μ. The field strength $F_{\alpha\mu\nu}$ is the exterior derivative of $A_{\mu\nu}$. The lagrangian of this system can be written

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{12v^2}\, F_{\alpha\mu\nu}F^{\alpha\mu\nu} + \frac{e^2}{2}\,(A_{\mu\nu} - \partial_\mu B_\nu + \partial_\nu B_\nu)(A^{\mu\nu} - \partial^\mu B^\nu + \partial^\nu B^\nu) \\
&- \pi i\, A_{\mu\nu} \int_\Sigma d\sigma^a \wedge d\sigma^b\, \delta^4(x - z(\sigma))\, \frac{\partial z^\mu}{\partial\sigma^a}\frac{\partial z^\nu}{\partial\sigma^b} \\
&+ \mu_0 \int_\Sigma d^2\sigma\, \sqrt{det\ g}\ .
\end{aligned}
\tag{8}
$$

This lagrangian has the gauge invariance $\delta A_{\mu\nu} = \partial_\mu\chi_\nu - \partial_\nu\chi_\nu$, $\delta B_\mu = \chi_\mu$. A unitary gauge exists for which $B_\mu = 0$. Equation (8) is the dual lagrangian to (5). It was found by precisely the same technique used to find the Kramers-Wannier dual of a lattice field theory or spin system (see for example [21]). The duality of gauge-invariant fields in more than two dimensions was first understood by Wegner (who only considered Z_2 invariant lattice models. However, Wegner's method is essentially the same as that used for continuum abelian systems) [22]. The first explicit discussion in the literature of the duality of continuum gauge fields coupled to continuum scalar fields appears to be that of Sugamoto [23]. The coupling of antisymmetric tensor gauge fields to strings was first investigated by Kalb and Ramond [24]. Kalb and Ramond's model was rederived and argued to describe vortices in a relativistic superfluid by Lund and Regge [25]. A study of the radiation of goldstone bosons by global cosmic strings in the Kalb-Ramond formulation was made by Vilenkin and Vachaspati [26]. Nambu made some suggestions concerning quark confinement and strings by introducing a mass in the Kalb-Ramond model [27] and a model of electric strings in QCD was developed using a non-abelian generalization of antisymmetric tensor gauge fields on the lattice [28]. Such non-abelian fields were also discussed by Nepomechie [29]. It was shown that strings are confined by dynamical membranes in Kalb-Ramond theories with magnetic-monopole instantons, both analytically [30] and numerically [31].

The propagator of the massive antisymmetric tensor gauge field in the unitary gauge can be written

$$
S(x - y)^{\mu\nu;\alpha\beta} = \frac{v^2}{2(-\partial^2 + e^2v^2)}\,\left(\delta^{\mu[\alpha}\delta^{\beta]\nu} - \frac{2}{e^2v^2}\partial^{[\mu}\delta^{\nu][\alpha}\partial^{\beta]}\right)\delta^4(x - y)
\tag{9}
$$

5 The World-Sheet Action

The easiest way to find the world-sheet action is by taking unitary gauge $B_\mu = 0$ and integrating out $A_{\mu\nu}$. A string tension will be induced by virtue of the exponential fall-off of the propagator [27], [28]. However, the full induced action has not heretofore been worked out.

Before proceeding further, it is neccesary to say a bit about the geometry of two-dimensional manifolds embedded in R^4. Define the antisymmetric tensor field on the world-sheet

$$t^{\mu\nu} = \frac{1}{\sqrt{2\,det\,g}}(e_1^\mu e_2^\nu - e_2^\mu e_1^\nu) \quad , \tag{10}$$

which will be called the tangent plane field. The tangent plane field was first discussed by Balachandran et. al. [12] and by Polyakov [6]. Notice that this field satisfies $t_{\mu\nu}t^{\mu\nu} = 1$. There are two four-component normal vectors n_k, $k = 3, 4$, satisfying $n_k \cdot e_a = 0$ and $n_k \cdot n_j = \delta_{kj}$. The derivative of the tangent vector e_a is given by

$$\partial_a e_a = \Gamma_{ab}^c e_c + K_{ab}^k n_k \quad , \tag{11}$$

where Γ_{ab}^c is the usual affine connection and K_{ab}^k is the second fundamental form or extrinsic curvature tensor.

The tangent plane field satisfies

$$\partial_a t^{\alpha\beta} = \frac{1}{\sqrt{2\,det\,g}}(K_{a1}^k n_k^\mu e_2^\nu - K_{a2}^k n_k^\mu e_1^\nu - K_{a1}^k n_k^\nu e_2^\mu + K_{a2}^k n_k^\nu e_2^\mu) \quad , \tag{12}$$

and by virtue of $K_{ab}^k = K_{ba}^k$,

$$\partial_\alpha t^{\alpha\beta} = \sum_\alpha g^{ac} e_c^\alpha \partial_a t^{\alpha\beta} = 0 \quad . \tag{13}$$

The euclidean space R^4 can be reparametrized by the new coordinates σ^1, σ^2, Ω^3 and Ω^4

$$x^\mu = x^\mu(\sigma, \Omega) = z^\mu(\sigma) + \Omega^k n_k^\mu(\sigma) \quad , \tag{14}$$

though such a parametrization does not assign unique coordinates to each point of R^4. The manifold consisting of the disjoint union of a covering of R^4 defined in this way is called the normal bundle of the manifold Σ. The normal bundle has base space Σ and each fiber is isomorphic to R^2. This parametrization has been used before by Gervais and Sakita for strings with small penetration depth and coherence length [5] and by Mazur and Nair in their discussion of QCD string actions [32].

Points in R^4 which are close enough to Σ (i.e. for which $(\Omega^3)^2 + (\Omega^4)^2$ is sufficiently small) can be uniquely parametrized by (14), and the metric in the new coordinates is

$$G_{ab}(\sigma, \Omega) = \sum_\mu \frac{\partial x^\mu}{\partial \sigma^a}\frac{\partial x^\mu}{\partial \sigma^b} \;, \;\; G_{ak}(\sigma, \Omega) = \sum_\mu \frac{\partial x^\mu}{\partial \sigma^a}n_k^\mu \;, \;\; G_{kj}(\sigma, \Omega) = \delta_{kj} \quad . \tag{15}$$

Notice that on the base manifold Σ, the 2×2 sub-block G_{ab} of this metric reduces to the induced metric, $G_{ab}(\sigma, 0) = g_{ab}$. The delta function of R^4 in these coordinates is

$$\delta^4(x^\mu(\sigma, \Omega) - x^\mu(\xi, \Pi)) = \frac{1}{\sqrt{det\,G}}\delta^2(\sigma - \xi)\,\delta^2(\Omega - \Pi) = \frac{1}{\sqrt{det\,g}}\delta^2(\sigma - \xi)\,\delta^2(\Omega - \Pi) \quad . \tag{16}$$

which is needed to obtain the propagator (9) in the new coordinate system, σ^1, σ^2, Ω^3, Ω^4.

One more formula is needed to obtain the world-sheet action in managable form. For any differential operator A which commutes with Ω^3 and Ω^4,

$$\int d^2\Omega \, \delta^2(\Omega) \, (-\sum_{k=3}^{4} \frac{\partial^2}{\partial(\Omega^k)^2} + A)^{-1} \, \delta^2(\Omega) = \frac{1}{4\pi} \int_0^\infty du \, (u + A)^{-1} \ . \qquad (17)$$

However, the right-hand-side is ultraviolet divergent. The divergence can be regularized by introducing an ultraviolet cut-off through a subtraction procedure. If M is some large number with the dimensions of inverse centimeters, the regularized version of (17) is

$$REG \quad [\int d^2\Omega \, \delta^2(\Omega) \, (-\sum_{k=3}^{4} \frac{\partial^2}{\partial(\Omega^k)^2} + A)^{-1} \, \delta^2(\Omega)]$$

$$= \quad \frac{1}{4\pi} \int_0^\infty du \, [(u + A)^{-1} - (u + M^2 + A)^{-1}] = \frac{1}{4\pi} \, \log \frac{A + M^2}{A} \ . \qquad (18)$$

The world-sheet action obtained from (8) is

$$I[z] = \frac{\pi^2}{4} \int_\Sigma d^2\sigma \int d^2\Omega \int_\Sigma d^2\xi \int d^2\Pi \sqrt{det \, g(\sigma)} \, t^{\mu\nu}(\sigma) \, \delta^2(\Omega)$$

$$\times \quad S(x(\sigma, \Omega) - x(\xi, \Pi))^{\mu\nu;\alpha\beta} \sqrt{det \, g(\xi)} \, t^{\alpha\beta}(\xi) \, \delta^2(\Pi) \ . \qquad (19)$$

This form, equation (19), of the world-sheet action is essentially analogous to the action discussed by Rasetti and Regge, Davis and Shellard and Lund for superfluid helium [17]. Using (9) together with (16) for the propagator, as well as (13) and (18), this simplifies to

$$I[z] = \frac{\pi v^2}{4} \int_\Sigma d^2\sigma \sqrt{det \, g} \, t^{\mu\nu} [\log(1 - \frac{\Delta}{\Lambda^2}) - \log(1 - \frac{\Delta}{e^2 v^2})] \, t_{\mu\nu}$$

$$+ \quad [\mu_0 + 4\pi v^2 \log(\frac{\Lambda^2}{e^2 v^2})] \int_\Sigma d^2\sigma \sqrt{det \, g} \ , \qquad (20)$$

where Δ is the covariant Laplacian,

$$\Delta = \frac{1}{\sqrt{det \, g}} \, \partial_a \, g^{ab} \sqrt{det \, g} \, \partial_b \ , \qquad (21)$$

and Λ is defined by $M^2 = \Lambda^2 + e^2 v^2$. The action (20), which is the main result of this paper, will henceforth be referred to as the Nielsen-Olesen action. It is not a local action, for it is non-polynomial in derivatives.

Physically, one can understand Λ as cutting off the high-momentum modes of the spin-1 boson field (formally, this has been viewed as the dual antisymmetric tensor field, rather than as a vector field). Thus, it performs the same function as the higgs mass. Recall that the role of the higgs field is to "soften" the high-energy exchange of massive spin-1 bosons. The cut-off Λ is the inverse coherence length, ξ^{-1}, and its role, as far as vortices are concerned, is the same as that of a finite-mass higgs field.

In the quantum theory there is another term which must be added to (20); the Liouville action, which has no extrinsic curvature dependence [33].

The validity of the Nielsen-Olesen action (20) obtained above breaks down as soon as any eigenvalue of the matrix K^k exceeds the reciprocal of the penetration depth $D^{-1} = ev$. If this is the case, the normal bundle is not locally isomorphic to R^4 a distance D away from the surface Σ.

Using (12), the Nielsen-Olesen action (20) can be expanded in powers of the second fundamental form. To leading order

$$
\begin{aligned}
I[z] \;\approx\; & -\frac{\pi v^2}{4} \, (\frac{1}{e^2 v^2} - \frac{1}{\Lambda^2}) \int_\Sigma d^2\sigma \, \sqrt{det\, g} \sum_k g^{ab} g^{ed} K^k_{ab} K^k_{ed} \\
& + \; [\mu_0 + 4\pi v^2 \log(\frac{\Lambda^2}{e^2 v^2})] \int_\Sigma d^2\sigma \, \sqrt{det\, g} \; ,
\end{aligned}
\tag{22}
$$

hence the string appears to be *anti-rigid* at this order. One might therefore guess that the Nielsen-Olesen action is unstable. It will be shown, at least for a special case, that the Nielsen-Olesen action really is stable, so long as the penetration depth D is greater than the coherence length ξ. However, no general proof of stability will be given here. Even for actions with a positive quadratic term in K, some simple classical solutions are known to be unstable [13]. In any case, (22) is useless, though (20) is not.

In the next section it will be shown that a wave instability occurs if the penetration depth $D = (ev)^{-1}$ is smaller than the coherence length, ξ, which is equal to the inverse cut-off Λ^{-1}. Otherwise, the Nielsen-Olesen action is stable, at least around large flat world-sheets. If $D < \Lambda^{-1}$, the vortex core, with $< \phi >= 0$, begins to fluctuate throughout the volume, destroying the condensate, where $|\phi| = v$. Thus, the superconductor type-II can no longer tolerate the entry of magnetic flux and becomes type-I. There is a line of phase transitions in the D vs. Λ^{-1} plane at $D = \Lambda^{-1}$, seperating these two regimes of superconductivity.

The string tension diverges in the limit of infinite Λ (just as the classical electron self-energy diverges if the electron radius is set to zero). Therefore, in this limit, the Nielsen-Olesen action is dominated by the Nambu term.

In references [7], [8], [9], the ratio of the coherence length and the penetration depth is fixed, leaving a single length scale, ϵ. Perturbation theory is then done about $\epsilon = 0$. The reader can see that at $\Lambda = \infty$ and $e^2 v^2 = \infty$ there is no extrinsic curvature dependence. If one takes $\Lambda = \epsilon^{-1}$ and $ev = (C\epsilon)^{-1}$ and expands the Nielsen-Olesen action (20), the result is essentially (22).

6   Stability of Flat Surfaces

While the question of stability will not be completely settled here, it appears likely that the Nielsen-Olesen action is stable, unless the penetration depth is less than the coherence length. In this section this will be shown for the special case of a flat surface.

By the implicit function theorem, it is always possible (at least locally) to impose the gauge $z^1 = \sigma^1$, $z^2 = \sigma^2$. A nearly flat surface, can be described by

$$
z^1 = \sigma^1 \; , \;\; z^2 = \sigma^2 \; , \;\; z^3 = f^3(\sigma) \; , \;\; z^4 = f^4(\sigma) \; .
\tag{23}
$$

Defining the Fourier transform of the fluctuation field, f, by

$$f^k(\sigma) = \int \frac{d^2p}{(2\pi)^2} \, f^k(p) \, e^{-i\,p\cdot\sigma} \quad, \tag{24}$$

the Nielsen-Olesen action (20) is, to quadratic order

$$I[z] = const. + \frac{\pi v^2}{8} \int \frac{d^2p}{(2\pi)^2} \sum_k f^k(p)f^k(-p) \, p^2 \, [\log \frac{p^2 + \Lambda^2}{p^2 + e^2v^2} + 2\mu_0] \quad, \tag{25}$$

The frequency of the fluctuations is therefore positive for all p, provided that the coherence length $\xi = \Lambda^{-1}$ is less than the penetration depth $D = (ev)^{-1}$. Otherwise, there is an instability signaling the onset of type-I superconductivity.

7 The Branched Polymer Phase of the Abelian Higgs Model

An obvious question is whether the abelian higgs model has a non-trivial string, with finite tension, after renormalization. It will be argued here that this is not the case. The conclusion is perhaps not surprising, for a theory which is not asymptotically free in the ultraviolet and which is likely to be infrared free. The argument hinges on the fact that the action of a world-sheet tube monotonically decreases as a function of the radius.

The main limitation of the arguments presented in this section is that the Liouville action [33] is not being considered. However, the Liouville action has no extrinsic curvature dependence, and is unlikely to stabilize the string.

Consider a tube of radius r. Take $\sigma^1 \epsilon [0, 2\pi r)$ and $\sigma^2 \epsilon (-\frac{L}{2}, \frac{L}{2})$ where $L >> r$. The world-sheet is then described by

$$z^1(\sigma) = \cos \frac{\sigma^1}{r} \;, \;\; z^1(\sigma) = \sin \frac{\sigma^1}{r} \;, \;\; z^3(\sigma) = \sigma^2 \;, \;\; z^4(\sigma) = 0 \;\;. \tag{26}$$

The area of the world-sheet is $A = 2\pi r L$. The action of this configuration can be worked out from (20). The result is

$$I[z] = \mu(r)A = [\mu_0 + \frac{\pi v^2}{4} \, \log(\frac{\Lambda^2 + r^{-2}}{e^2 v^2 + r^{-2}})]A \;\;. \tag{27}$$

Notice that the action of a thin tube (it should be assumed that $r > D$ for the Nielsen-Olesen action (20) to be valid) is proportional to its length. Therefore, in the euclidean quantum field theory, the usual energy versus entropy arguments for random surfaces [11] imply that branched polymers dominate the world-sheet. This means that the quantum abelian higgs model vortex-world-sheet is dominated by branched-polymer configurations, and the vortex string tension will not scale in the limit that the cut-off is removed.

Not all the details of the quantization of the world-sheet have been taken into account in this section. In particular, the Liouville action has been ignored. However, it appears unlikely that the dominance of branched polymers will be affected by the inclusion of the Liouville action. Therefore the Nielsen-Olesen vortex almost certainly does not exist as a quantum soliton of the field theory.

8 Discussion

In this paper, the full world-sheet action of a Nielsen-Olesen vortex in an abelian higgs model was obtained, where the effect of a finite higgs mass was approximated by the introduction of an ultraviolet cut-off. In particular, the extrinsic curvature dependence was found. The action was argued to be stable, provided that the penetration depth is greater than the coherence length. It is almost certainly the case that the vortex world-sheets of the quantum field theory are dominated by branched polymer configurations, implying that no scaling limit exists in which the renormalized string tension is finite.

Acknowledgements

Note Added

The world-sheet action of a non-relativistic vortex can be obtained using similar methods. After this paper appeared as a preprint, the author discovered that a number of people had recently studied such Abrikosov vortex strings using dual Kalb-Ramond fields [34], [35]. In particular, the action analogous to (8) (originally found in [17]) has been obtained [35] in this way. The authors of reference [35] intentionally dropped certain terms containing time derivatives, which were not needed for their purposes. Including these terms, the author has reduced the action for a nonrelativistic vortex to an action in which the dependence on the curvature of the vortex is made explicit, using the normal-bundle parametrization discussed here. It can then be shown that for this case as well, flat surfaces are stable provided the coherence length is less than the penetration depth.

Recently, Sato and Yahikozawa calculated the extrinsic curvature dependence of the relativistic Nielsen-Olesen world-sheet to second order in K, obtaining a result similar to (22) [36]. I would like to thank them for discussions concerning their work.

References

[1] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61** (1973) 45.

[2] A. A. Abrikosov, Soviet Phys. - JETP **5** (1957) 1174; J. Phys. Chem. Solids **2** (1957) 925.

[3] Z. F. Ezawa and C. H. Tze, Nucl. Phys. **B100** (1975) 1; J. Math. Phys. **17** (1976) 2278; Phys. Rev. **D14** (1976) 1006.

[4] D. Förster, Nucl. Phys. **B81** (1974) 84.

[5] J. L. Gervais and B. Sakita, Nucl. Phys. **B91** (1975) 301.

[6] A. M. Polyakov, Nucl. Phys. **B268** (1986) 406.

[7] K. I. Maeda and N. Turok, Phys. Lett. **B202** (1988) 376.

[8] R. Gregory, Phys. Lett. **206B** (1988) 199.

[9] R. Gregory, D. Haws and D. Garfinkle, Phys. Rev. **D42** (1990) 343; R. Gregory, Phys. Rev. **D43** (1991) 520.

[10] L. Peliti and S. Leibler, Phys. Rev. Lett. **54** (1985) 1690; W. Helfrich, J. Physique **46** (1985) 1263; D. Förster, Phys. Lett. **A114** (1986) 115.

[11] B. Durhuus, J. Fröhlich and T. Jónsson, Nucl. Phys. **B240** (1984) 453.; J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. **B270** (1986) 433; and P. Orland, Nucl. Phys. **B275** (1986) 457.

[12] A. P. Balachandran, F. Lizzi and G. Sparano, Nucl. Phys. **B263** (1986) 608.

[13] E. Braaten and C. K. Zachos, Phys. Rev. **D35** (1987) 1512.

[14] T. W. B. Kibble, J. Phys. **A9** (1976) 1387; A. Vilenkin and E. P. Shellard, **Topological Defects and Cosmology**, Cambridge University Press, Cambridge, MA (1993).

[15] M. J. Bowick, L. Chandar, E. A. Schiff and A. M. Srivastava, Science **263** (1994) 943.

[16] M. I. Polikarpov, U. J. Weise and M. A. Zubkov, Phys. Lett. **B309** (1993) 133.

[17] M. Rasetti and T. Regge, Physica **80A** (1975) 217; L. Davis and E. P. S. Shellard, Phys. Rev. Lett. **63** (1989) 2021; F. Lund, Phys. Lett. **A159** (1991) 245.

[18] H. Arodz and P. Wegrzyn, Phys. Lett. **B291** (1992) 251.

[19] S. I. Tomonaga, Prog. Theor. Phys. **5** (1950) 544; D. Bohm and D. Pines, Phys. Rev. **75** (1952) 338; **92** (1953) 92; **102** (1956) 1189; D. Bohm and B. Salt, Rev. Mod. Phys., **39** (1967) 894; A. Jevicki and B. Sakita, Nucl. Phys. **B165** (1980) 511.

[20] T. W. Kirkman and C. K. Zachos, Phys. Rev. **D25** (1981) 999.

[21] R. Savit, Rev. Mod. Phys. **19** (1979) 1198.

[22] F. Wegner, J. Math. Phys. **12** (1971) 2259; see also R. Balian, J. M. Drouffe and C. Itzykson, Phys. Rev. **D11** (1975) 2098.

[23] A. Sugamoto, Phys. Rev. **D19** (1979) 1820.

[24] M. Kalb and P. Ramond, Phys. Rev. **D9** (1974) 2273.

[25] F. Lund and T. Regge, Phys. Rev. **D14** (1976) 1524.

[26] A. Vilenkin and T. Vachaspati, Phys. Rev. **D35** (1987) 1138.

[27] Y. Nambu. Phys. Rev. **D10** (1974) 4262.

[28] P. Orland, Phys. Lett. **B122** (1983) 78; in **Gauge Theory on a Lattice: 1984**, Proc. of the Argonne Nat. Lab. Workshop, C. Zachos, W. Celmaster, E. Kovacs and D. Silverman, eds., NTIS, Springfield, VA (1984); Imperial College preprint (1984).

[29] R. I. Nepomechie, Nucl. Phys. **B212** (1983) 301.

[30] P. Orland, Nucl. Phys. **B205** (1982) 107.

[31] R. Pearson, Phys. Rev. **D26** (1982) 2013.

[32] P. O. Mazur and V. P. Nair, Nucl. Phys. **B284** (1986) 146.

[33] A. M. Polyakov, Phys. Lett. **103B** (1981) 207.

[34] A. Zee, Nucl. Phys. **B421** (1994) 111.

[35] M. Hatsuda, S. Yahikozawa, P. Ao and D. J. Thouless, Phys. Rev. **B49** (1994) 15870.

[36] M. Sato and S. Yahikozawa, Kyoto preprint, June 1994.